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Stochastic Hydrology

Random functions

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Introduction

Up to now: *unknown value* of a variable *Z* without considering time and spatial co-ordinates *t*,*x*,*y*,*z*
→ Random Variable

Now consider the *unknown variation* of a variable *Z* in time, space or space-time: and spatial co-ordinates Z(*t*), Z(*x*,*y*,*z*) or Z(*x*,*y*,*z*,*t*)
→ Random Function

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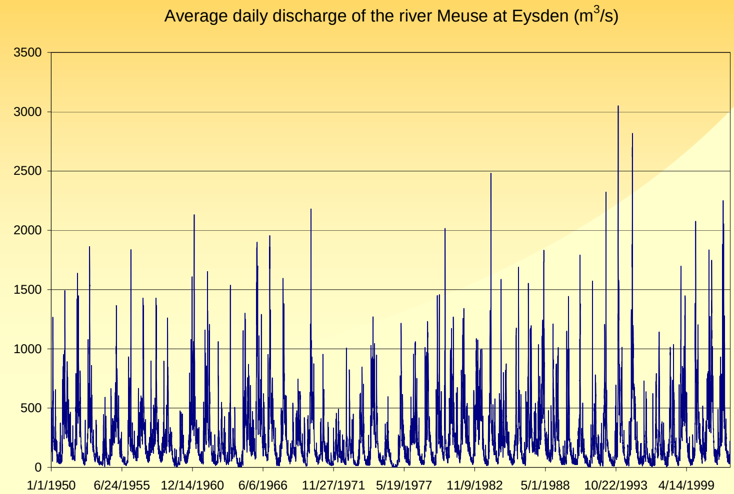
Z(t):



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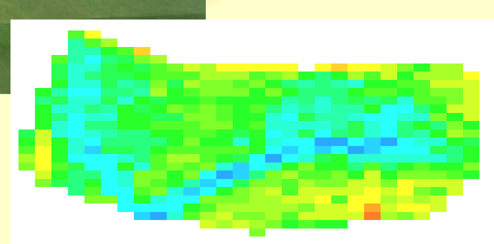
Z(x,y) :



Exhaustive TDR data set (2000 observations) Tarrawarra Catchment Australia

(Western and Grayson, 1998, Water Resources Research, 34(10): pp2765-2768)

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Exhaustive TDR data set (2000 observations) Tarrawarra Catchment Australia

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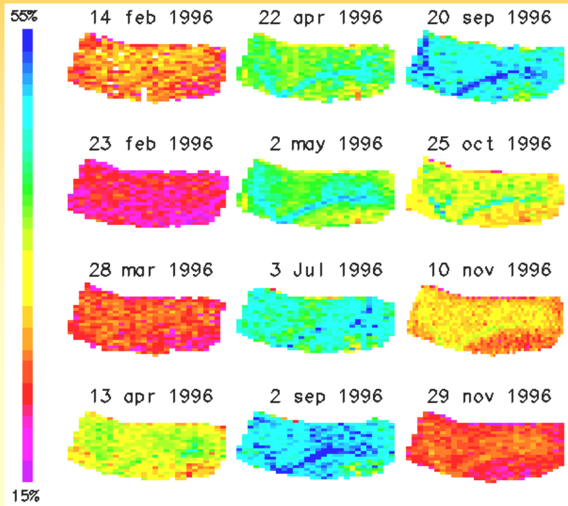
$Z(x,y,t)$:



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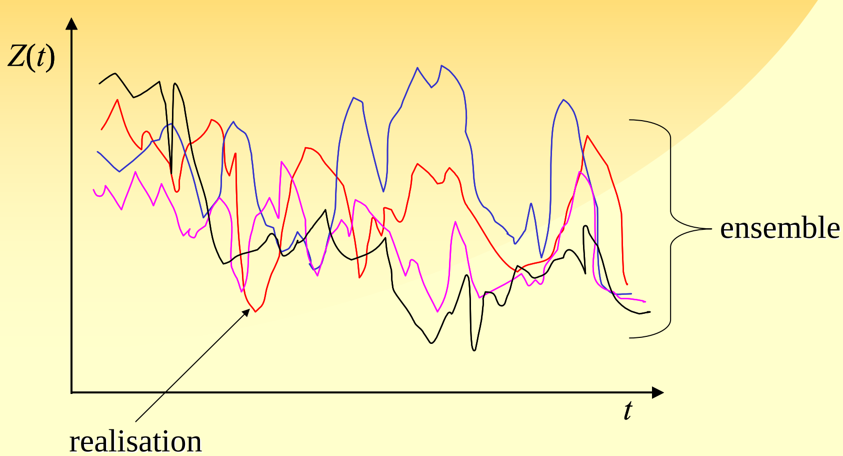
Concept of a random function



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Random functions of different domains

- Random time function (RTF), random process or stochastic process:

$$Z(t)$$

- Random space function or random field:

$$Z(\mathbf{x}), \mathbf{x} = (x, y) \text{ or } \mathbf{x} = (x, y, z)$$

- Random space-time function or space-time random field:

$$Z(\mathbf{x}, t)$$

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Random functions are dependent RVs

Another way of viewing a random function is as:

a collection of random variables (one at every location in space or point in time) that are all mutually statistically dependent.

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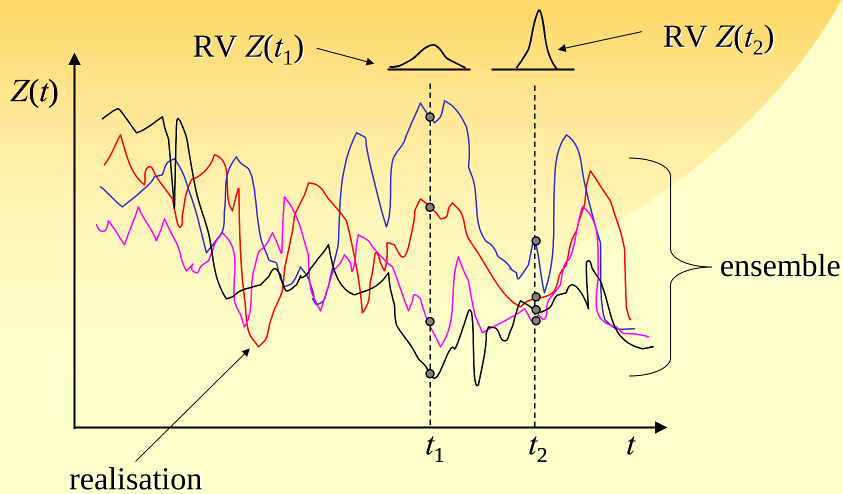
Random functions are dependent RVs



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Univariate Statistics of random functions

Mean:

$$\mu_Z(t) = \mathrm{E}[Z(t)] = \int_{-\infty}^{\infty} z\, f_Z(z;t)dz$$

Variance:

$$\sigma_Z^2(t) = \mathrm{E}[\{Z(t) - \mu_Z(t)\}^2] = \int_{-\infty}^{\infty} \{z - \mu_Z(t)\}^2 f_Z(z;t)dz.$$

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Bivariate Statistics of random functions

Bivariate pdf

$$f(z_1, z_2; t_1, t_2) = \lim_{\partial z_1, \partial z_2 \to 0} \frac{\Pr[z_1 < Z(t_1) \le z_1 + \partial z_1, z_2 < Z(t_2) \le z_2 + \partial z_2]}{\partial z_1 \partial z_2}$$

Gives the probability density of values of the random function at two points in time, space or space-time.

Covariance:

$$\mathrm{COV}[Z(t_1), Z(t_2)] = \mathrm{E}[\{Z(t_1) - \mu_Z(t_1)\}\{Z(t_2) - \mu_Z(t_2)\}] =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{z_1 - \mu_Z(t_1)\}\{z_2 - \mu_Z(t_2)\} f(z_1, z_2; t_1, t_2) dz_1 dz_2$$

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Multivariate pdf of a random function

Also called: multi-point pdf

$$f(z_1, z_2, \ldots.\, z_N \,;\, t_1, t_2, \ldots.\, t_N) =$$

$$\lim_{\partial z_1, \ldots, \partial z_N \to 0} \frac{\Pr(z_1 < z(t_1) \le z_1 + \partial z_1, z_2 < z(t_2) \le z_2 + \partial z_2, \ldots, z_N < z(t_N) \le z_2 + \partial z_N}{\partial z_1 \partial z_2 \cdots \partial z_N}$$

Gives the probability density of values of the random function at a set of points in time, space or space-time.

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Types of random functions

1) Value: a) discrete valued D

   b) continuous valued Z

2) Domain type: a) time t

   b) space $\mathbf{x}$

   c) space-time $(\mathbf{x},t)$

3) Domain definition: a) continuous $\mathbf{x}$, t

   b) discrete (at finite number of points) $\mathbf{x}_i$, t_k

   c) random points (point process) $\boldsymbol{X}_j$, T_k

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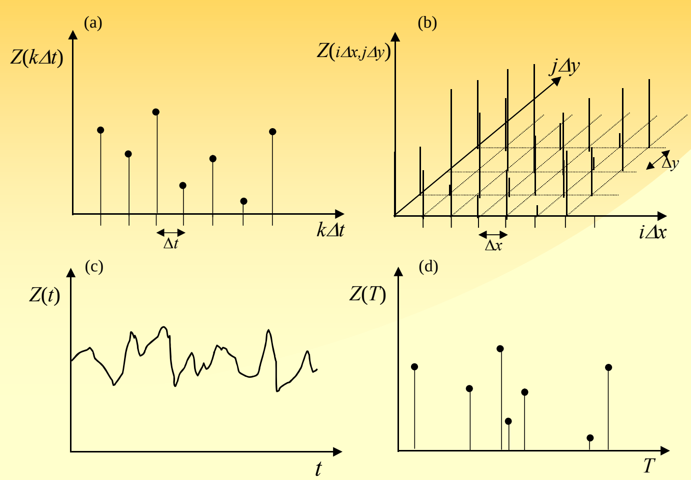
Types of random functions



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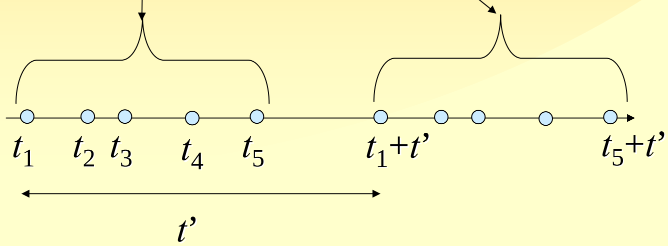
Strict stationary random functions

$$f(z_1, z_2, \ldots, z_N;\ t_1, t_2, \ldots\ t_N) =$$
$$f(z_1, z_2, \ldots, z_N\ ;\ t_1 + t', t_2 + t', \ldots,\ t_N + t') \quad \forall t_i, t'$$



time

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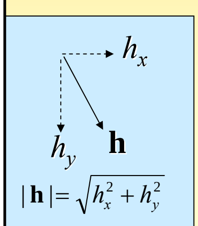
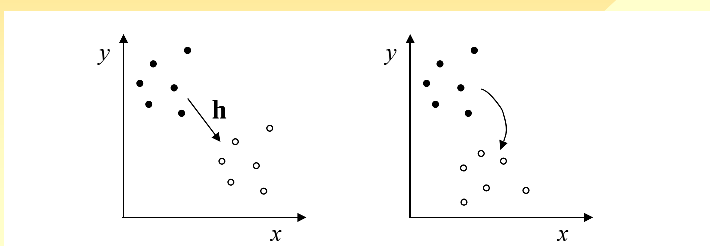
Strict stationary random functions

$$f(z_1, z_2, \ldots, z_N;\ \mathbf{x}_1, \mathbf{x}_2, \ldots\ \mathbf{x}_N) =$$
$$f(z_1, z_2, \ldots, z_N\ ;\ \mathbf{x}_1 + \mathbf{h}, \mathbf{x}_2 + \mathbf{h}, \ldots,\ \mathbf{x}_N + \mathbf{h}) \quad \forall \mathbf{x}_i, \mathbf{h}$$



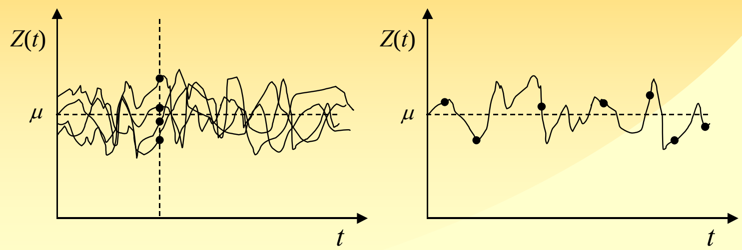


Strict stationary RF

Strict Stationary and isotropic RF

space

Ergodic random functions



Statistics over time (space/space-time) converge to the statistics over the ensemble!
WE CAN THUS ESTIMATE STATISTICS FROM DATA!

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Ergodic random functions

e.g. ergodicity in the mean:

$$\lim_{T\to\infty}\frac{1}{T}\int_T z(t)dt = \int_{-\infty}^{\infty} z\, f_Z(z;t)dz = \mu_Z$$

Requirements:

- random function is stationary
- time interval/area is large compared to correlation scale

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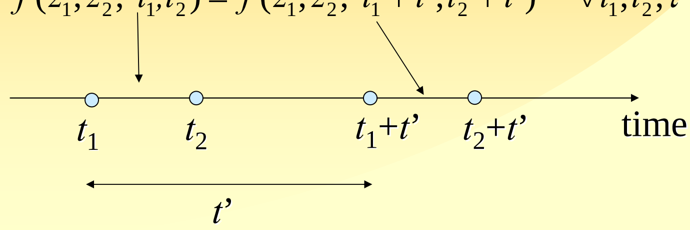
Second order stationary RF

Bivariate pdf invariant under translation:

$$f(z_1, z_2;\ t_1, t_2) = f(z_1, z_2;\ t_1 + t', t_2 + t') \quad \forall t_1, t_2, \tau$$



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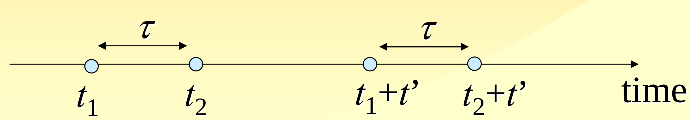
Wide sense stationary RF

Time:

$\mu_Z(t) = \mu_Z$ is constant

$\sigma_Z^2(t) = \sigma_Z^2$ is constant and finite

$\text{COV}[Z(t_1), Z(t_2)] = C_Z(t_2 - t_1) = C_Z(\tau)$



Covariance function $C(\tau)$

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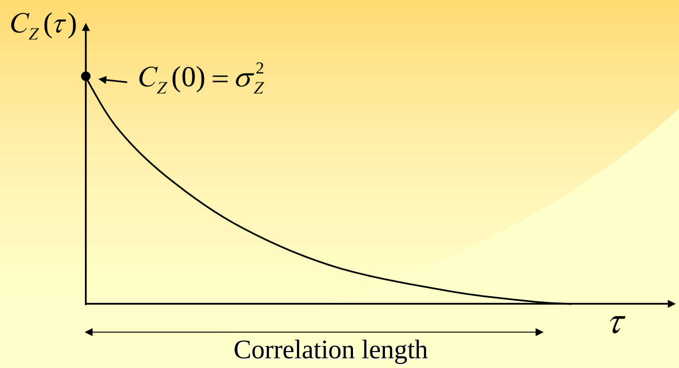
Covariance function



Note: $C(\tau) = C(-\tau)$

time

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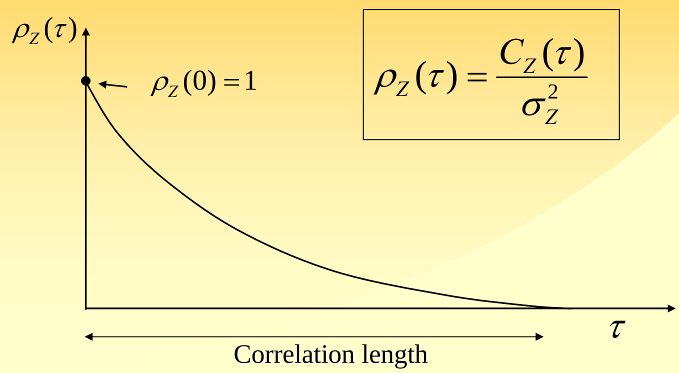
Correlation function



$$\rho_Z(\tau) = \frac{C_Z(\tau)}{\sigma_Z^2}$$

time

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Stochastic Hydrology

Random functions (2)

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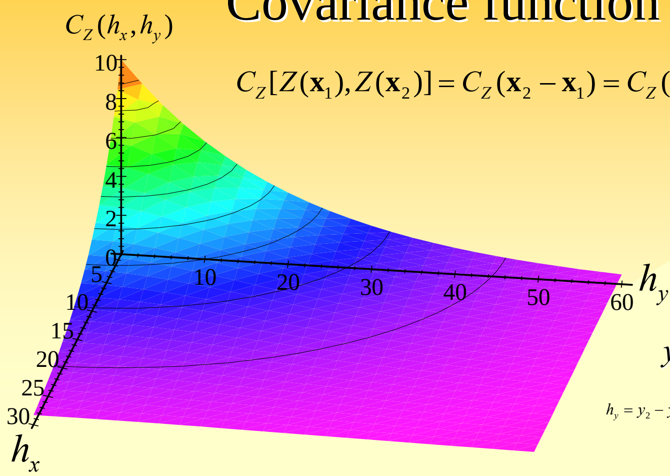
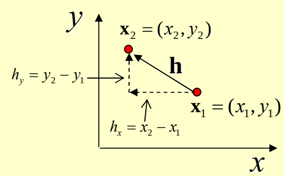
space

Covariance function

$$C_Z[Z(\mathbf{x}_1), Z(\mathbf{x}_2)] = C_Z(\mathbf{x}_2 - \mathbf{x}_1) = C_Z(\mathbf{h}) = C_Z(h_x, h_y)$$





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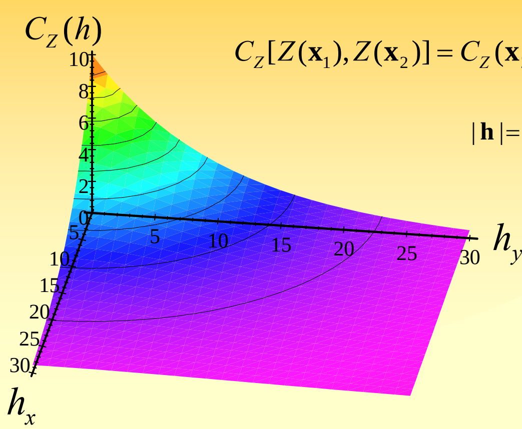
space

Covariance function

$$C_Z[Z(\mathbf{x}_1), Z(\mathbf{x}_2)] = C_Z(\mathbf{x}_2 - \mathbf{x}_1) = C_Z(|\mathbf{h}|) = C_Z(h)$$

$$|\mathbf{h}| = h = \sqrt{h_x^2 + h_y^2}$$



Isotropy: both correlation scales are the same

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Estimation of Covariance functions

Time

$$\hat{C}_Z(k\Delta t) = \frac{1}{n-k}\sum_{i=1}^{n-k}(z_i - \hat{\mu}_Z)(z_{i+k} - \hat{\mu}_Z)$$

Space

$$\hat{C}_Z(\mathbf{h}) = \frac{1}{n(\mathbf{h})}\sum_{i=1}^{n(\mathbf{h})}[z(\mathbf{x}) - \hat{\mu}_Z][z(\mathbf{x}_i + \mathbf{h} \pm \Delta\mathbf{h}) - \hat{\mu}_Z]$$

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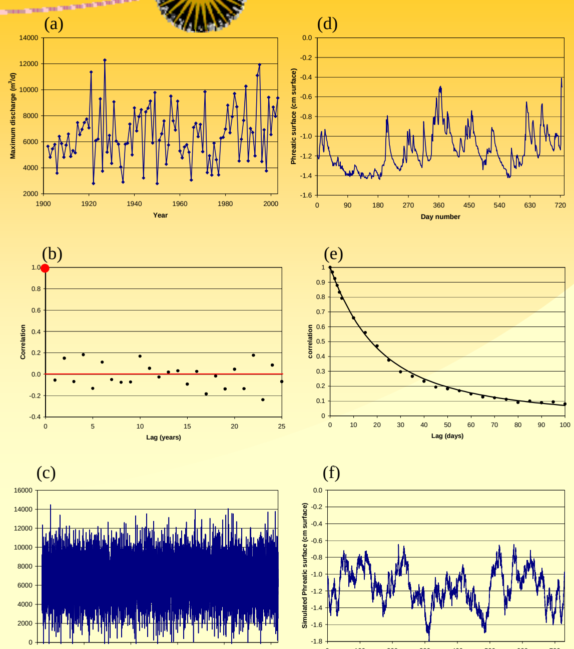
Influence of correlation



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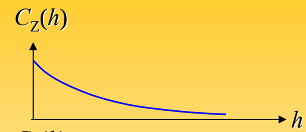
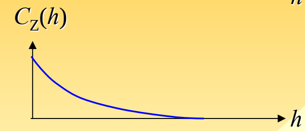
Covariance functions

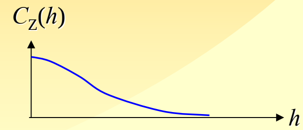
Exponential



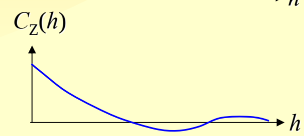
Spherical



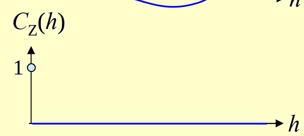
Gaussian



Hole effect (wave)



White noise



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Measures of correlation

Time and space

- Correlation length or (effective) range:

  Lag τ at which $\rho(\tau)=0$ (spherical) or $\rho(\tau)<0.05$ (exponential)

Time

- Integral scale:

$$I_{Z(t)} = \int_0^\infty \rho(\tau)\,d\tau$$

- Scale of flucuation:

$$\theta = \int_{-\infty}^{\infty} \rho(\tau)\,d\tau = 2I_{Z(t)}$$

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Measures of correlation

Space (2D and 3D)

- Integral scale:

2D: $$I_{Z(\mathbf{x})} = \left[\frac{4}{\pi} \int_0^\infty \int_0^\infty \rho(h_1, h_2) dh_1 dh_2 \right]^{1/2}$$

3D: $$I_{Z(\mathbf{x})} = \left[\frac{6}{\pi} \int_0^\infty \int_0^\infty \rho(h_1, h_2, h_3) dh_1 dh_2 dh_3 \right]^{1/3}$$

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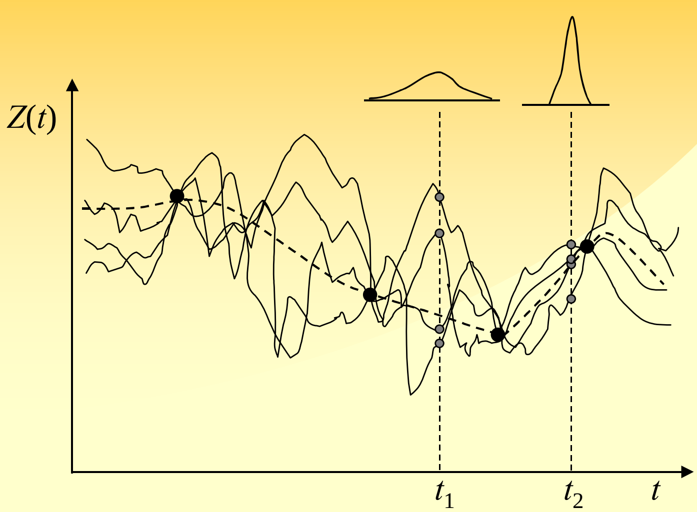
Conditional random functions



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Conditional random functions

Conditional random function: $Z(t \mid y_1,..,y_m)$

Conditional density function at $t=t_1$: $f_Z(z;t_1 \mid y_1,..,y_m)$

Conditional mean and variance $t=t_1$:

$$\mu_{Z|Y} = E[Z(t) \mid y_1,..,y_m] = \int_{\infty}^{\infty} f_Z(z;t_1 \mid y_1,..,y_m)dz$$

$$VAR[(t \mid y_1,..,y_m)] = \int_{\infty}^{\infty} (z-\mu_{Z|Y})^2 f_Z(z;t_1 \mid y_1,..,y_m)dz$$

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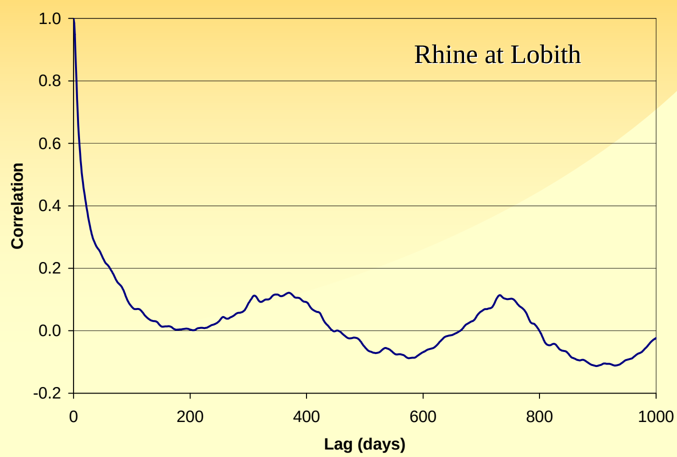
Spectrum

Many hydrological time series have a seasonal character



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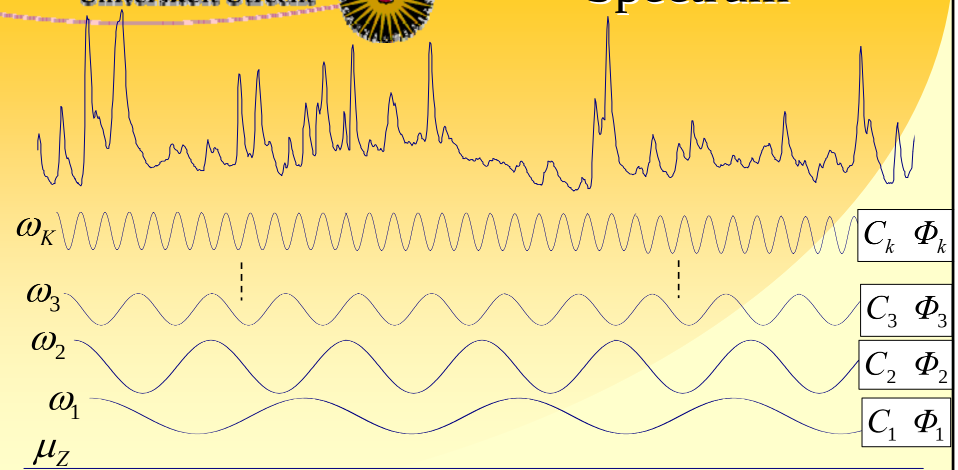
Spectrum



$$Z(t) = \mu_Z + C_1 \cos(\omega t + \Phi_1) + C_2 \cos(2\omega t + \Phi_2) + \ldots + C_K \cos(K\omega t + \Phi_K)$$

Stationary random functions

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Spectrum

Then we have: $\sigma_Z^2 = \int_0^\infty G_Z(\omega) d\omega$

$G_Z(\omega)$: (one-sided) spectral density function

$G_Z(\omega)$: The magnitude of the harmonic signal with wavelength ω in the process Z(t)

$$\sigma_Z^2 \sim \sum_i Var[C_k]$$

Spectrum

Spectrum and covariance are related:

$$C_Z(\tau) = \int_0^\infty G_Z(\omega)\cos(\omega\tau)\,d\omega$$

$$G_Z(\omega) = \frac{2}{\pi}\int_0^\infty C_Z(\tau)\cos(\omega\tau)\,d\omega$$

Wiener-Khinchine relations

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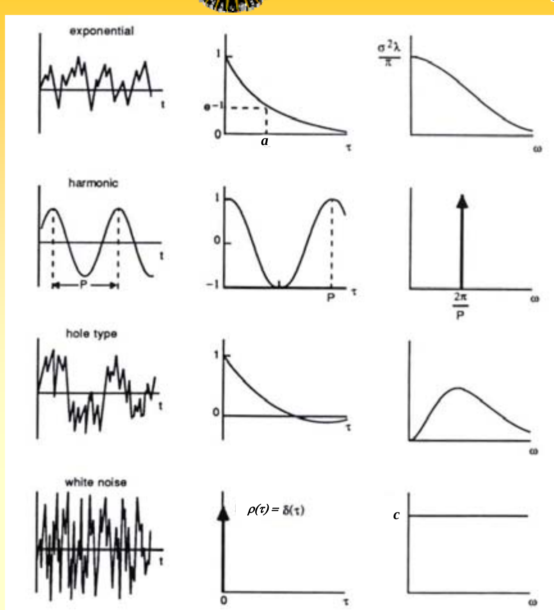
Spectrum



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Spectrum

Spectrum can be used for:

- Analysis of periodicity in data
- Filtering (leve out certain wavelengths)
- Solving stochastic differential equations

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Spectrum Rhine data set